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Laplace transform and its applications into dynamic systems: a review

Transformada de Laplace y sus aplicaciones en sistemas dinámicos: una revisión

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RESUMEN

Este artículo presenta una revisión sobre las aplicaciones de la transformada de Laplace (TL) en el análisis de sistemas dinámicos lineales. Se presenta la importancia de la transformada; la definición de la transformada; los teoremas y propiedades con aplicaciones y por último, aplicaciones del uso de la TL para el cálculo de: error de estado estacionario, funciones de transferencia, solución de ecuaciones diferenciales ordinarias, respuestas en frecuencia, análisis de señales y diagramas de bloques. Se propone, como situación didáctica para propósitos académicos, el modelamiento y análisis operacional con TL de un reactor multipropósito.

ABSTRACT:

This paper presents a review of Laplace Transform (LT) applications in the analysis of linear dynamic systems. Is presented the importance of the transforms; the definition of the Transform; theorems and properties with applications and finally, the use of LT for the calculation of: steady state error, transfer functions, ordinary differential equations solution, frequency responses, signal analysis and block diagrams. The modeling and operational analysis with LT of a multipurpose reactor is proposed as a didactic situation for academic purposes.

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1. Introduction

A physical phenomenon can be modeled taking into account the physics laws through mathematical equations, according to them: linear and non-linear (polynomial and algebraic); and equations in derivatives -linear or non-linear (Ordinary or Partial), that is, considering functions that change in time.

On the other hand, dynamic systems explain such phenomena. These can be classified into mechanical, electrical, thermal, hydraulic, among others, according to signals, their treatment and context; and can be represented in different ways: using ordinary or partial differential equations; establishing transfer functions; constructing state equations; developing block diagrams; calculating frequency responses, among others [1]-[5].

In addition, LT is the main mathematical tool used for signal analysis, solving ordinary or partial differential equations, developing models and control systems. Therefore, it becomes a fundamental basis for the dynamic systems analysis; hence, it is important to understand its definition, existence and uniqueness, as well as the theorems and transformed functions or basic signals [4], [6]-[8].

This is why, given the importance of the LT for the dynamic systems analysis, it is of interest in engineering documentary investigations that lead to a revision of the subject that evidences that - in spite of being a not very recent study area - it is still used in diverse applications of dynamic systems but different from those that are traditionally approached academically or in lines of research in development.

As a result, this paper is organized as follows: Section 2 sets out the documentary research methodology. Section 3 describes the relevance of Laplace Transform, the definition of Laplace Transform and the main theorems. Section 4 describes the relevance of Laplace Transform into dynamic systems. To illustrate the most common applications of LT in Dynamic Systems, Section 5 is established. Section 6 shows an application example in a multipurpose reactor; and finally, Section 7 presents the conclusions.

2. Materials and methods

In order to establish the methodology developed in this documentary research, the synthetic method is used[9] . This method analyzes and synthesizes the collected information, which allows structuring the ideas throughout the review on Dynamic Systems that apply the Laplace Transform as a tool to know its main characteristics.

Likewise, for the exploratory study, the bibliographic search used the following databases: IEEE Xplore, ScienceDirect, Google Scholar; as well as class texts. The references obtained range from 60 to 70, with the keywords used corresponding to the categories: Laplace Transform, Dynamic Systems, Classical Control, Control Systems, Differential Equations. The endorsement of this methodology was obtained from the experts of the Research Group on Order and Chaos (ORCA) of the Universidad Distrital Francisco José de Caldas. The Index Method was used for the construction of the revision (vertebrate from a general index) [10], [11]..

3. Definition and Properties of Laplace Transform

The Laplace transform of a function *f (t)* with domain in the real variable t is equivalent to another function that depends on a new variable s in the complex domain. The Laplace transform of *f (t)*, for $y>0$, is defined by [5], [8], [12]-[21]

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt \quad (1)$$

The Laplace transform of *f (t)* exists if the integral (1) converges; the adequate conditions of existence are: continuity by sections and the exponential order for *f(t)* [5], [8], [14]-[21].

3.1. Theorems of the Laplace Transform

Elementary functions are transformed from the definition; however, it is not convenient to use the Laplace transform definition to find more complex function transforms, since the integration process is often complicated. Based on the bibliographic review carried out, Table 1 and Table 2 was developed [4] [8], [14]-[29] which presents in a detailed and clear way the transformed theorems most applied in dynamic systems.

Direct Laplace Transform	
Definition	**Example**
1. First Translation Theorem	
All expression of the form $\mathcal{L}[f(t)e^{\pm at}]$ will be equal to the transformed function $f(t)$, replacing $s \rightarrow s \mp a$ $$\mathcal{L}[\, e^{\pm at}\, f(t)] = F(s \mp a) \qquad (2)$$	Find the Laplace Transform of: $$f(t) = e^{-at}\, sen\, at\ \ \mu(t) \qquad (3)$$ Applying the complex translation theorem in (3) and identifying the function $f(t)$ and evaluating $s \rightarrow s + a$, the transform of (3) is: $$F(s) = \frac{a}{(s+a)^2 + 1} \qquad (4)$$
2. Second Translation Theorem	
Any expression of the form $f(t-a)\mu(t-a)$, its Laplace transform will be $\mathcal{L}[f(t)]$ multiplied by e^{-as}, where a represents the displacement value. $$\mathcal{L}[g(t)] \equiv G(s) = \ e^{-as}F(s) \qquad (5)$$	Find the Laplace Transform of: $$f(t) = [sen\, t\, \mu(t + 2\pi)] \qquad (6)$$ Applying the displacement property over time in (6) identifying $f(t) = \text{sen}\, t$ and $a = 2\pi$, as $\text{sen}(\, t\ + 2\pi) = \text{sen}\, t$ the transform of (6) is: $$F(s) = \frac{1}{s^2+1} e^{2\pi s} \qquad (7)$$
3. Multiplication by t Theorem	
If $f(t)$ is a sectionally continuous function and exponential order then every Laplace transform: $\mathcal{L}\{t\, f(t)\}, \mathcal{L}\{t^2 f(t) \ldots\}$, converges on s. That is, the Laplace transform of the product of a function $f(t)$ with t^n can be found by differentiating the Laplace transform from $f(t)$. $$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n} \quad for\ n = 1,2,3, \ldots \qquad (8)$$	Find the Laplace Transform of: $$f(t) = \{t^2\ \ sen\, t\}\, \mu(t) \qquad (9)$$ Applying the theorem and identifying $f(t)$ in (9) its transform is: $$F(s) = \frac{6s^2 - 2}{(s^2+1)^3} \qquad (10)$$
4. Derivative Theorem (real differentiation)	
Leaving the function $f(t)$ continuous in pieces and having a continuous derivative up to order n-1 $f^{(n-1)}(t)$ and sectionally continuous derivable $f^{(n)}(t)$ in an always finite interval, the transformed $f^{(n)}(t)$ exists if $t \geq 0$ $$\mathcal{L}\left[\frac{d^n f(t)}{df^n}\right] = \ s^n F(s) - s^{n-1} f(0) - f^{(n-1)}(0) \qquad (11)$$	Find: $$\frac{dx}{dt} + 3x = \mu(t) \qquad (12)$$ Applying the Derivative Theorem in (12) its transform is: $$sX(s) - X(0) + 3X(s) = \frac{1}{s} \qquad (13)$$

5. Integral Theorem (real integration)

If $f(t)$ is sectionally continuous, the expression $\int_0^t f(\alpha)\,d\alpha$ has Laplace transform given by:

$$\mathcal{L}\left[\int_0^t f(\alpha)\,d\alpha\right] = \frac{f(s)}{s} + \frac{1}{s}f^{-1}(0) \quad (14)$$

Find the Laplace Transform of:

$$f(t) = \left[\int_0^x t\, e^{at}\sin t\ dt\right]\mu(t) \quad (15)$$

Identifying the function that is being integrated, and applying the corresponding theorems its transform is:

$$\mathcal{L}\{t\, e^{at}\sin t\} = \frac{2(s-a)}{((s-a)^2+1)^2} \quad (16)$$

Applying the Laplace Integral Theorem, (16) is multiplied by $1/s^n$, where n is the integral number, the transform of (15) is:

$$F(s) = \frac{2(s-a)}{s\,((s-a)^2+1)^2} \quad (17)$$

6. Integral Transform Theorem (complex integration)

If $f(t)$ is transformable into Laplace and having that $\lim_{t\to 0}\frac{f(t)}{t}$ exists, the function integral $\int_s^\infty F(s)\,ds$ corresponds to the Laplace transform division of a function $f(t)$ given by:

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\ ds \quad (18)$$

If and only if $\lim_{t\to 0}\frac{f(t)}{t}$ exists.

Find the Laplace Transform of:

$$f(t) = \left[\frac{\sin t}{t}\right]\mu(t) \quad (19)$$

The $f(t)$ limit exist, then the complex integration theorem applies. The variable t indicates the number of integrals to apply.

$$\int_s^\infty \frac{1}{s^2+1}\,ds = \tan^{-1} s|_s^\infty \quad (20)$$

The transform of (19) is:

$$F(s) = \frac{\pi}{2} - \tan^{-1} s \quad (21)$$

7. Initial Value Theorem

Leaving $f(t)$ being transformable function, then in the case of evaluating $\lim_{s\to\infty} sF(s)$, exists.

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0+} f(t) \quad (22)$$

Determine initial value of:

$$x(0) = \frac{3s^2+4s+1}{s^3+2s^2+s+2} \quad (23)$$

Applying Initial Value Theorem:

$$x(0) = \lim_{s\to\infty} s\,\frac{3s^2+4s+1}{s^3+2s^2+s+2} \quad (24)$$

Evaluating the limit:

$$x(0) = 3 \quad (25)$$

8. Final Value Theorem	
Leaving $f(t)$ being transformable function, then in the case of evaluating $\lim_{t\to\infty} f(t)$, exists. $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ (26)	Determine final value of: $x(0) = \frac{3s^2+4s+1}{s^4+3s^3+3s^2+2s}$ (27) Applying Final Value Theorem: $x(0) = \lim_{s\to 0} s\frac{3s^2+4s+1}{s^4+3s^3+3s^2+2s}$ (28) Evaluating the limit: $x(0) = \frac{1}{2}$ (29)
9. Convolution Theorem	
If functions $f(t)$ and $g(t)$ are sectionally continuous at $[0,\infty)$, then a special product, denoted as $f(t) * g(t)$, is defined as: $\int_0^t g(t-\tau)f(\tau)\,d\tau$ (30) The transformation multiplication represents the function convolution $f(t)$ y $g(t)$, defined as: $\mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)$ (31)	Find the Laplace Transform of: $f(t) = \left[\int_0^t e^t \sin(t-\tau)\,d\tau\right]\mu(t)$ (32) Applying Convolution Theorem and applying transform to $f(t)$ and $g(t)$ of (32), the result is: $F(s) = \frac{1}{(s-1)(s^2+1)}$ (33)
10. Frequency Convolution Theorem – s plane	
The Laplace transform of the product of two functions by steps and sectionally continuous $f_1(t)$ and $f_2(t)$ corresponds to the Convolution of its transformed ones. $\mathcal{L}[f_1(t)f_2(t)] = \frac{1}{2\pi j}F_1(s) * F_2(s)$ (34)	Find the Laplace Transform of: $f(t) = e^{-t}e^{-2t}\mu(t)$ (35) Applying Frequency Convolution Theorem, the transform is: $F(s) = \frac{1}{s+3}$ (36)

Table 1.Direct Laplace Transform Theorems.Source: own.

Factor	Factor Form	Partial Fraction Form
0	$A = constant$	Does not exist
1	$(ax+b)$	$\frac{A}{ax+b} \to Constant. A\ to\ be\ determined$

1	$(ax+b)^n$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}+\cdots$
3	(ax^2+bx+c)	$\frac{Ax+B}{ax^2+bx+c}$
3	$(ax^2+bx+c)^n$	$\frac{Ax+B}{(ax^2+bx+c)}+\frac{Cx+D}{(ax^2+bx+c)^2}+\frac{Ex+F}{(ax^2+bx+c)^3}+\cdots$
3	(ax^3+bx^2+cx+d)	$\frac{Ax^2+Bx+C}{ax^3+bx^2+cx+d}$
3	$(ax^3+bx^2+cx+d)^n$	$\frac{Ax^2+Bx+C}{(ax^3+bx^2+cx+d)}+\frac{Dx^2+Ex+F}{(ax^3+bx^2+cx+d)^2}+\cdots$

Table 2. Useful partial fractions to calculate the inverse Laplace transform [5], [7], [8], [16], [18], [22], [30].

Table 2.Useful partial fractions to calculate the inverse Laplace transform [5], [7], [8], [16], [18], [22], [30].

3.2. Importance of the LT

Dynamic systems model various physical phenomena through differential equations; the classical way of exerting control over these systems is through the Laplace transform. Its application varies in several fields[31].

Specifically speaking of control, Laplace has been used in the simulation and control of a large number of systems: pendulums [32], here Laplace is used to obtain system models; thermoelectric systems [33], where the thermal regulation system of a transformer is analyzed in Laplace's domain and its response in time is found with the inverse Laplace transform; among others, where the Laplace transform is used to obtain frequency domain equations that ease the controllers design [34]_[36].

In addition, Laplace's methods have been applied to compensate in the response of digital control systems [37]; signal models for digital controllers [38]; and although today there are several innovative control techniques, Laplace's transform is still a tool that facilitates the systems understanding [39], [40]. What is important is that, based on these analyses, new control models are formed, such as algorithms based on the uncertainty and disturbance estimation (UDE) [41]. In another way, it has been applied to the containment control of multi-agent systems where its collaborative control is analyzed from the perspective of the frequency domain that LT offers [42]. Laplace transforms produce efficient models to describe the dynamics of other types of systems [43], as shown by [44] where adequate current control is sought; now, by means of LT, a frequency domain technique is applied for the design of multivariable control systems to passive beam vibration control [45]. On the other hand, the Laplace transform is not only a design tool, it is also a tool for verifying the control systems stability [46]-[48], viability [49], redesigning them [50], and for finding answers to inputs in dynamic systems [51], [52].

In addition to control systems, LT serves as an analytical tool [53], for the development of models and calculation methods [54]-[56]. In [57] it is applied to filters to mitigate low frequency voltage pulsations and calculate harmonics in direct current. In electrical and electronics the uses of LT are diverse [58]-[60]. Methods have been developed to determine the transient and dynamic stresses arising near the fracture system in elastic bodies under deformation, this is based on both direct and inverse Laplace transform methods '[61]. Furthermore, new formulas have been proposed for the transient matrix of soil resistance to an overhead power line; each formula was obtained using inverse Laplace transform, its use produced a time-domain solution of electromagnetic transients in multiconductor lines [62]. Similarly, LT has been applied in transmission line models described in the s (frequency) domain to obtain transient voltage and current profiles along multiconductor lines [63]. LT also has a place in complex dynamic systems such as those of varied times in which its bidimensional version is used [64]; as well as in modeling studies for graphene sheets where Laplace helps to obtain tangential current density parameters through Maxwell's laws [65].

On the other hand, the inverse Laplace Transform - whose goal is to bring what is in the frequency domain into the time domain- has also been a tool in multiple mathematical analyses that seek to obtain a time-domain response in various fields of study. For example, [66] shows the inverse transform use in canonical electromagnetic wave equations. In the development of a new design method for plasmonic antennas that generate localized circular polarized light for recording at height [67], where having the magnetic field equations in a complex domain, the inverse Laplace transform is used to obtain the answer in the time domain.

Finally, LT in fields other than control and electronics has made its appearance in variants such as the study of CO_2 and its atmospheric changes in response to emissions [68]. In the same line, it has been applied in the selection of biological materials to determine its characteristics and differentiations [69]; and it has been used to convert the dynamic equations of a linear system of equations into applications to the method of intervals for analysis of structures [70].

4. Importance of LT in dynamic systems

As it has been seen, LT has been used in signal analysis, solution of differential equations, linear models and control systems, among others. But the present exploration of Dynamic Systems has been made from mathematical models [1]-[5]. Figure 1 shows a diagram where from the equations that model the system, the Laplace transform allows to obtain other characteristic parameters of the system, such as the transfer function, the block diagrams, the representation of states, the analysis in frequency and at the same time how they interact among them.

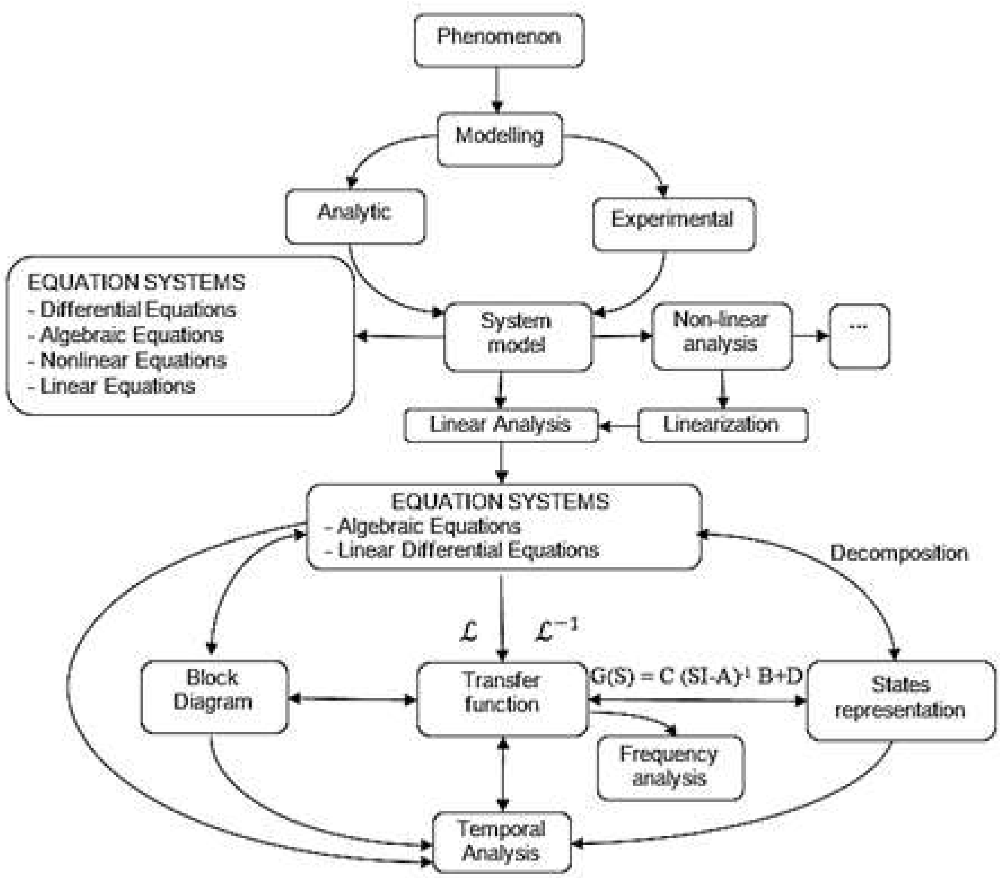

Figure 1. Laplace and dynamic systems interaction diagram. Source: own.

5. Applications of LT in dynamic systems

As it was studied previously, the Laplace Transform in the dynamic systems allows to find main characteristics of these models, within these processes the application in analysis of signals, solution of differential equations, analysis of electrical models, calculation of transfer function, representation of block diagrams, calculation of steady state error and analysis of frequency response is highlighted [1][5], [71]-[73]. Figure 2 shows a methodology to make use of the transformed in the analysis of dynamic systems.

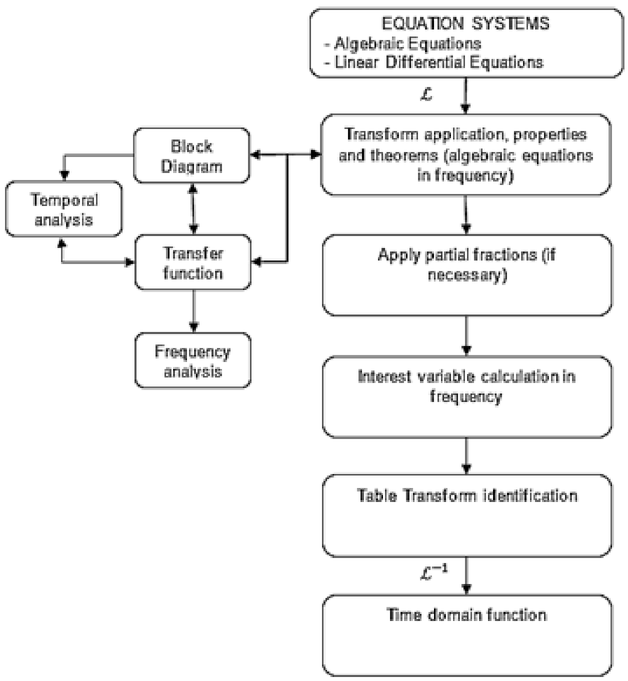

Figure 2. Methodology for the Laplace Transform application in dynamic systems. Source: own.

5.1. Signal analysis

Find the LT of signal Figure 3 [24].

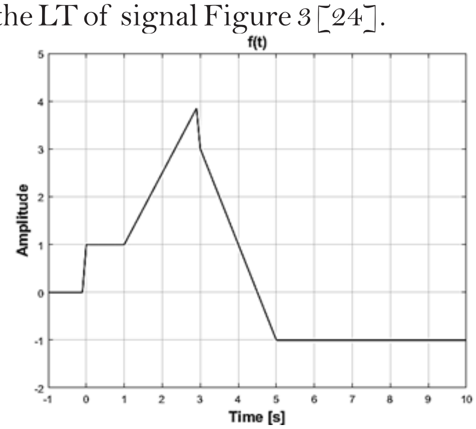

Figure 3. Mixed signal of step and ramp functions. Source: own

The function described in Figure 3 is given by:

$$f(t) = [\mu(t) + 1.5\, r(t-1) - \mu(t-3) - 3.5\, r(t-3) + 2\, r(t-5)\,] \quad (37)$$

Applying the property of linearity and displacement in time Table 1, the signal (37) transform is:

$$\frac{1}{s} + 1.5\,\frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s} - 3.5\,\frac{e^{-3s}}{s^2} + 2\,\frac{e^{-5s}}{s^2} \quad (38)$$

5.2. Ordinary Differential Equations

Find the solution to the differential equation equation [23], [24] :

$$Y'' + 2Y' + 5Y = e^{-t}\sin t \qquad Y(0) = 0, \quad Y'(0) = 1 \quad (39)$$

Applying Laplace transform in (39) Table 1, is obtained

$$\{s^2 y - sY(0) - Y'(0)\} + 2\{sy - Y(0)\} + 5y = \frac{1}{(s+1)^2 + 1} \quad (40)$$

Solving (40) and replacing initial conditions

$$y(s) = \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} \quad (41)$$

Applying inverse Laplace transform Table 2, the response time of (41) is

$$Y(t) = \frac{1}{3} e^{-t}(\sin t + \sin 2t) \quad (42)$$

5.3. Electrical Circuit Analysis

Apply Laplace transform to the equation that models the following system :

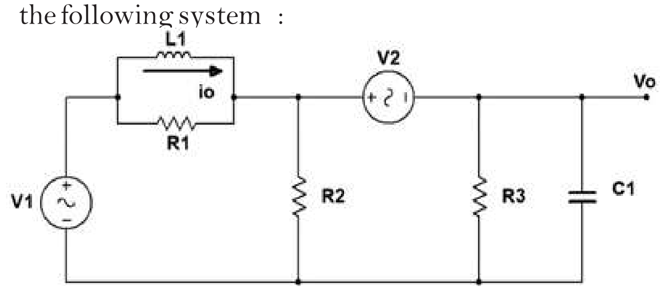

Figure 4 . Electrical system. Source: own.

The equations that model the system in Figure 4 are

$$\frac{dVc}{dt} = \frac{1}{C}\left(-V_c\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) + I_L + \frac{V_{i1}}{R_1} - V_{i2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right) \quad (43)$$

$$\frac{dI_L}{dt} = \frac{1}{L}\left(V_{i1} - (V_{i2} + V_c)\right) \quad (44)$$

$$V_o = V_c\ , \quad I_o = I_L \quad (45)$$

If the system conditions are::

$$V_{i1} = u(t-10)\,, \quad V_{i2} = 2\,, \quad I_L(0) = 1A\,, \quad Vc(0) = 0\,, \quad R_1 = R_2 = R_3 = 1\Omega\,, \quad L = 2H\,, \quad C = 1F \quad (46)$$

Applying initial conditions and Laplace Transform Table 1 to (43), (44) and (45) is obtained:

$$sVc(s) = -3V_c(s) + I_L(s) + V_{i1}(s) - 2V_{i2}(s) \quad (47)$$

$$sI_L(s) - I_L(0) = \tfrac{1}{2}\left(V_{i1}(s) - V_{i2}(s) - V_c(s)\right) \quad (48)$$

$$v_o(s) = v_c(s) \quad (49)$$

Clearing the value for V_o and I_o is obtained:

$$V_o(s) = \frac{\left(2 + V_{i1}(s)(2s+1) - V_{i2}(s)(4s+1)\right)}{[2s^2 + 6s + 1]} \quad (50)$$

$$I_o(s) = \frac{\left(V_{i1}(s)[s+2] - V_{i2}(s)[s+1] + 2(s+3)\right)}{[2s^2 + 6s + 1]} \quad (51)$$

To the output response as a time function is applied inverse Laplace transform Table 2 into (50), (51) obtaining:

$$V_o(t) = 0.3779e^{-(0.1771)t}u(t) - 0.3779e^{-(2.8228)t}u(t) - 0.6891e^{-(0.1771)t}u(t-10) \quad (52)$$
$$- 0.3110e^{-(2.8228)t}u(t-10) + 1u(t-10) + 0.6223e^{-(0.1771)t}u(t)$$
$$+ 1.3779e^{-(2.8228)t}u(t) + 2u(t)$$

$$I_o(t) = 1.0669e^{-(0.1771)t}u(t) - 0.0669e^{-(2.8228)t}u(t) \quad (53)$$

5.4. Transfer Function Calculation

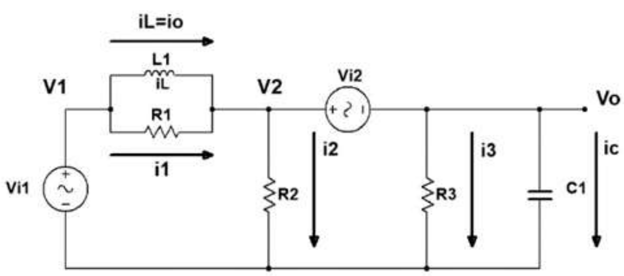

Figure . Electrical system. Source: own.

Based on Figure 5, the transfer function is calculated [24]. Eliminating initial conditions in (43) and (44), clearing the equations in terms of: $\frac{V_o}{V_{i1}}$, $\frac{V_o}{V_{i2}}$, $\frac{I_o}{V_{i1}}$, $\frac{I_o}{V_{i2}}$

$$C\frac{dV_o}{dt} = -V_o\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) + I_L + \frac{V_{i1}}{R_1} \quad (54)$$

$$L\frac{dI_L}{dt} = V_{i1} - V_o \quad (55)$$

Performing the corresponding operations, application of the Laplace transform Table 1, grouping similar terms the transfer functions of the circuit of Figure 5 are obtained.

$$\frac{V_o(s)}{V_{i1}(s)} = \left[\frac{(R_1 + Ls)(R_2R_3)}{R_1R_2R_3LCs^2 + LsR_2R_3 + LsR_1R_3 + LsR_1R_2 + R_1R_2R_3}\right] \quad (56)$$

The other transfer functions are calculated in the same way as found (56).

$$\frac{V_o(s)}{V_{i2}(s)} = \left[\frac{-(R_1R_2 + R_2Ls)(R_3)}{R_1R_2R_3LCs^2 + LsR_2R_3 + LsR_1R_3 + LsR_1R_2 + R_1R_2R_3}\right] \quad (57)$$

$$\frac{I_o(s)}{V_{i1}(s)} = \left[\frac{(R_2 + R_3 + R_2R_3R_1 + Cs)(R_1)}{R_1R_2R_3LCs^2 + LsR_2R_3 + LsR_1R_3 + LsR_1R_2 + R_1R_2R_3}\right] \quad (58)$$

$$\frac{I_o(s)}{V_{i2}(s)} = \left[\frac{-(1 + R_3Cs)(R_1R_2)}{R_1R_2R_3LCs^2 + LsR_2R_3 + LsR_1R_3 + LsR_1R_2 + R_1R_2R_3}\right] \quad (59)$$

5.5. Block Diagram Representation

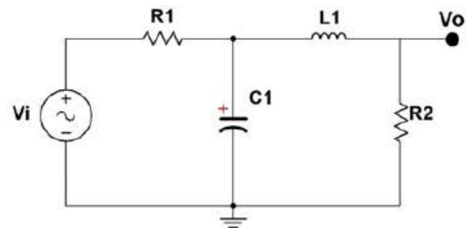

Figure 6. **Electrical system. Source: own.**

Initial system conditions :

$$V_i(t) = u(t),\ R_1 = 2\Omega,\ C = 3F,\ L = 4H, \quad (60)$$
$$R_2 = 3\Omega,\ V_C(0) = 1V,\ I_L(0) = 3A$$

The equations that model the system in Figure 6 are:

$$C\frac{dV_C}{dt} = \frac{V_1}{R_1} - \frac{V_C}{R_1} - i_L \quad (61)$$

$$L\frac{di_L}{dt} = V_C - R_2 \cdot i_L \quad (62)$$

$$V_0 = i_L \cdot R_2 \quad (63)$$

Applying Laplace transform into (61), (62) and (63) according to theorems Table 2, identifying the integrators number, in this case two given that there is more than one differential equation and the number of inputs and outputs.

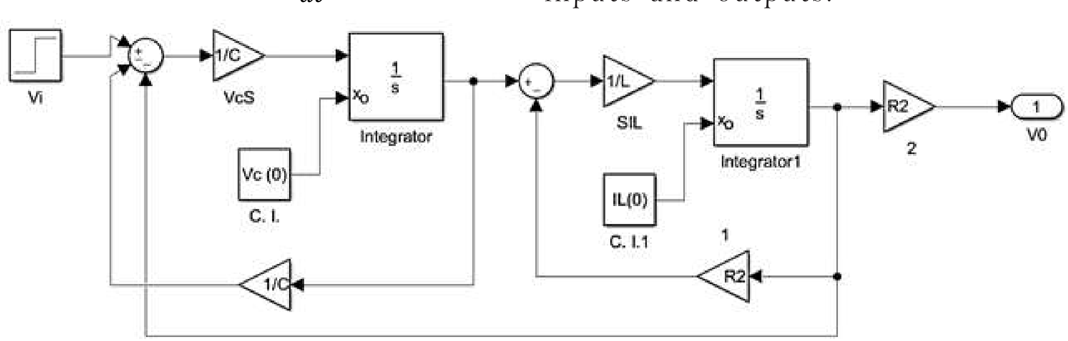

Figure 7 . Obtained block diagram. Source: own.

5.6. State-space representation to Transfer Function

the model state-space representation is given by (64) and (65)

$$\dot{x} = A * x + B * u \quad (64)$$

$$Y = C * x + D * u \quad (65)$$

Where x represents the state variables, u is the system input and the output A,B,C and D are arrays.
Based on the equations (61), (62) and (63) of Figure 6, the state-space representation is given by (66) y (67)

$$\begin{bmatrix} \dot{V_C} \\ \dot{I_L} \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1 * C} & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R_2}{L} \end{bmatrix} \begin{bmatrix} V_C \\ I_L \end{bmatrix} + \begin{bmatrix} \frac{1}{C * R_1} \\ 0 \end{bmatrix} V_i \quad (66)$$

$$V_O = \begin{bmatrix} 0 & R_2 \end{bmatrix} \begin{bmatrix} V_C \\ I_L \end{bmatrix} + 0 * V_i \quad (67)$$

The transfer function is found from the equation (68)

$$G(s) = C * (s * I - A)^{-1} * B + D \quad (68)$$

By applying equation (68) the transfer function is obtained

$$G(s) = \frac{R_2}{R_1 C L s^2 + (R_1 R_2 C - L)s + (R_1 - R_2)} \quad (69)$$

5.7. Frequency analysis

The frequency analysis for a second order system is characterized by having two components, the gain margin and phase, next, are shown the equations that allow to detail this behavior

$$G(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\xi\omega_n\omega + \omega_n^2} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\xi\frac{\omega}{\omega_n}} \quad (70)$$

Magnitude in dB.

$$|G(j\omega)|_{dB} = -10\log\left(\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\frac{\omega^2}{\omega_n^2}\right) \quad (71)$$

Phase in degree.

$$\phi(\omega) = -\tan^{-1}\left(\left(2\xi\frac{\omega}{\omega_n}\right) / \left(1 - \frac{\omega^2}{\omega_n^2}\right)\right) \quad (72)$$

Given the transfer function (73),

$$G(s) = \frac{(5s + 50)}{s^2 + 99.8s - 20} \quad (73)$$

Frequency analysis is performed as this comes from the Laplace transform calculation, the system magnitude response (74) and Figure 8.

$$|G(j\omega)|_{dB} = \frac{(5j\omega + 50)}{j\omega^2 + 99.8j\omega - 20} = -7.96 \quad (74)$$

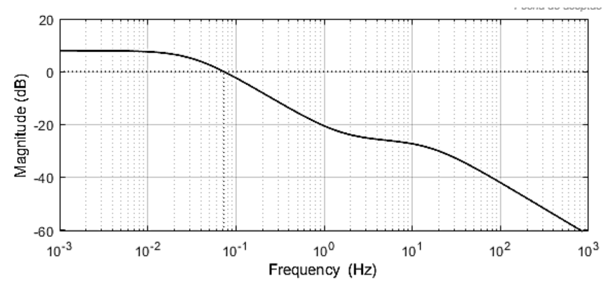

Figure 8 . System magnitude response. Source: own.

The system phase response (75) and Figure 9.

$$\phi(\omega) = \frac{(5j\omega + 50)}{j\omega^2 + 99.8j\omega - 20} = 68.8 \quad (75)$$

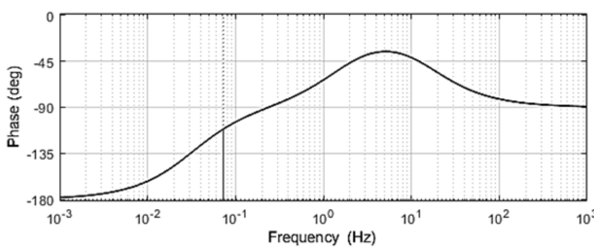

Figure 9 . System phase response. Source: own

From the analysis performed (74) and (75), allows to see the non-stability of the system.

5.8. Steady state error

One of the most common applications of this theorem in control systems is the calculation of steady state error, this is defined as system error to the difference between the reference signal and the input signal [24] .
The steady-state error calculation of position, velocity, and acceleration comes from the general formula found in the unit feedback system (76).

$$e(s) = \frac{r(s)}{1 + T(s)} \quad (76)$$

Applying the final value theorem Table 1, is obtained:

$$e_{ss} = \lim_{s \to 0} se(s) = \lim_{s \to 0} \frac{sR(s)}{1 + T(s)} \quad (77)$$

Where (77) corresponds to the system steady state error with unit feedback.
According to the above analysis, the equations for the steady state error calculation of position, velocity and acceleration are defined.

$$e_{ssp} = \lim_{s \to 0} \frac{1}{1 + T(s)} \quad (78)$$

$$e_{ssv} = \lim_{s\to 0} \frac{1}{ST(s)} \quad (79)$$

$$e_{ssa} = \lim_{s\to 0} \frac{1}{S^2T(s)} \quad (80)$$

With the controller and plant functions in Figure10, the system steady state error for position, speed and acceleration is analyzed.

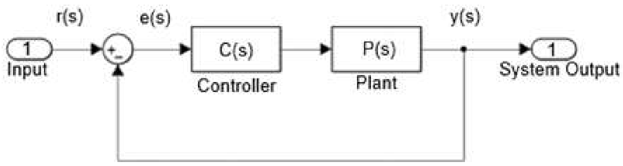

Figure 10 . Control system diagram. Source: own.

The controller and plant functions correspond to:

$$C(s) = \frac{2}{s(s+0.05)} \quad (81)$$

$$P(s) = \frac{s+0.1}{s^2(s^2+6s+3)} \quad (82)$$

The system's transfer function meets the formula:

$$T(s) = C(s) * P(s).$$

$$T(s) = \frac{2(s+0.1)}{s^3(s+0.05)(s^2+6s+3)} \quad (83)$$

Applying the final value theorem Table 1 the steady state error in position is:

$$e_{ssp} = \lim_{s\to 0}\left\{\frac{s^3(s+0.05)(s^2+6s+3)}{s^3(s+0.05)(s^2+6s+3)+2(s+0.1)}\right\} = 0 \quad (84)$$

In the same way, the speed and acceleration steady state error is obtained.

$$e_{ssv} = \lim_{s\to 0}\left\{\frac{s^2(s+0.05)(s^2+6s+3)}{2(s+0.1)}\right\} = 0 \quad (85)$$

$$e_{ssa} = \lim_{s\to 0}\left\{\frac{s(s+0.05)(s^2+6s+3)}{2(s+0.1)}\right\} = 0 \quad (86)$$

The steady state error analysis results for the system are shown in Table 3.

Model	e_{ssp} (%)	e_{ssy}(%)	e_{ssa}(%)
Type 1	0	0	0

Table 3. System steady state error. Source: own.

The system's response to step, ramp, and parabola inputs corresponds to Figure 11, Figure 12, and Figure 13.

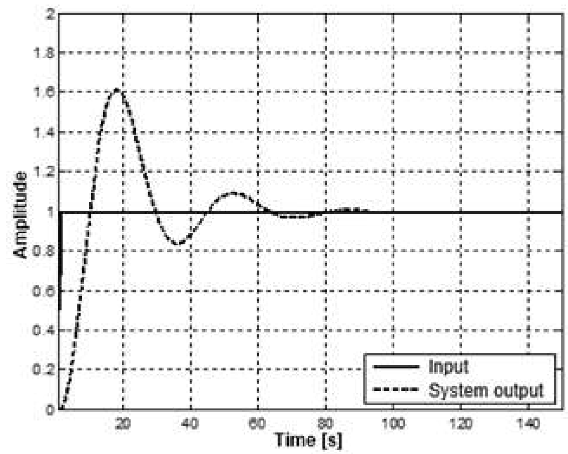

Figure 13 . Parabolic type input response. Source: own.

The process consists of a multipurpose reactor which was designed and implemented at the Universidad Distrital Francisco José de Caldas, in the control engineering curriculum project. This project was developed with the purpose of designing a PID reactor temperature controller [74].

The biodiesel manufacturing process begins by requesting the amount of raw material to be used, process temperature, dehydration temperature, washing decanting time, dehydration time and mixing speed, after doing this the plant will add the oil quantity entered by the user, when the reactor has the raw material level required, it will proceed to agitate and heat the oil between 55°C and 60°C required for the biodiesel production. Once the temperature is reached, the catalyst is added and a recirculation pump is activated to guarantee the mixture [74]. At this point the transesterification process begins and depends on the time input, then passes to the decantation time where the action of the PID controller and the agitator are turned off and the system remains in standby while this time happens, then starts the washing process where water enters the reactor responsible for dissolving methanol remains, sodium hydroxide and hydrosoluble components that will be eliminated later by decanting, after washing the decanting begins where water is separated with dissolved residues of biodiesel, finally passes to the stage of dehydration where the residual water is eliminated in biodiesel maintaining a high temperature to extract water vapors through a

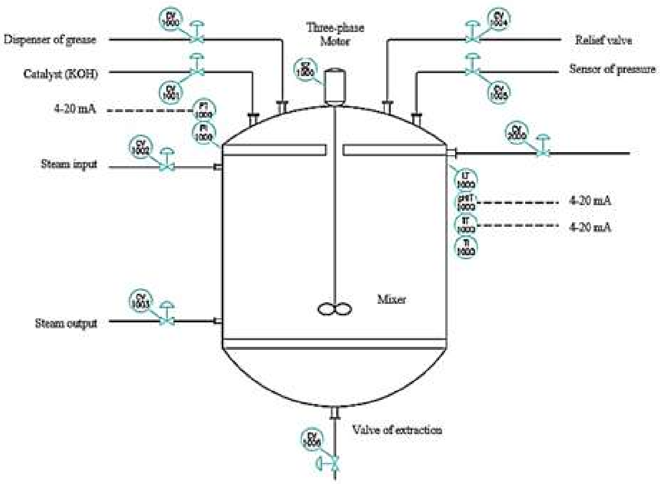

Figure 14 . Reactor P&ID diagram [74] .

It is evident that it is necessary to maintain temperature control for each stage of the process. The temperature is taken by a PT100, the linearization experiments were made between 17 °C and 86 °C [74] .

In order to design the controller it was necessary to obtain the plant transfer function and thus be able to analyze its dynamic behavior, the method to find it was to design 2 tests in which it can be compared the importance in the data amount to generate the model with the identification tool Ident of Matlab. The tests consist of injecting a step signal into the system to bring the plant to a stability point; it was obtained that the plant has a delay of 75 seconds and a stabilization time of approximately 4 hours, time in which the water reaches its boiling point, the first test has a sampling time of 30 seconds, while the second test has a sampling time of 1 second [74], from the measurements taken and using the identification tool Ident of Matlab, it is possible to establish the transfer function for each experiment, as a result the transfer function is taken thrown by the second test that has a model percentage of accuracy of 98.67%.

It should be noted that [74], made the identification in discrete time (z Transformed), therefore, from the measurement tables in this paper was developed the identification in continuous time LT, to reach the equation system that model the plant and see the behavior of it.

The transfer function thrown by the tool is:

$$F(s) = \frac{3{,}233\ x10^{-5}}{s + 4{,}633\ x10^{-5}} \qquad (87)$$

Based on the previous transfer function, the equations that model the reactor are known by applying the Laplace transform theorems Table 1 and the proposed application methodology Figure 5.

$$\frac{V_o(s)}{V_i(s)} = \frac{3{,}233\ x10^{-5}}{s + 4{,}633\ x10^{-5}} \qquad (88)$$

Clearing (88) in terms of the output function, and replacing the input with a step signal, is obtained:

$$V_o(s) = \left(\frac{1}{s}\right)\left(\frac{3{,}233\ x10^{-5}}{s + 4{,}633\ x10^{-5}}\right) \qquad (89)$$

Applying partial fractions in (89) Table 2 and inverse transform, is had:

$$V_o(t) = \left[0.69782 - 0.69782\ e^{-(4{,}633\ x10^{-5})t}\right]u(t) \qquad (90)$$

Based on the transfer function (87) the system block diagram is elaborated Figure 15.

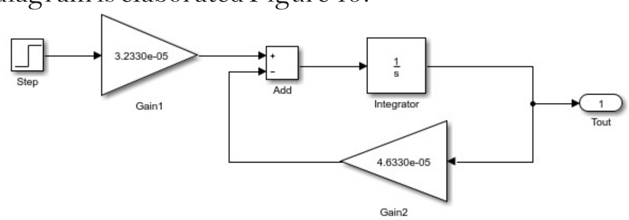

Figure 15 . Reactor block diagram. Source: own.

Frequency analysis is performed (87), the system magnitude response (91) and Figure 16.

$$|G(j\omega)|_{dB} = \frac{3{,}233\ x10^{-5}}{j\omega + 4{,}633\ x10^{-5}} = \infty \qquad (91)$$

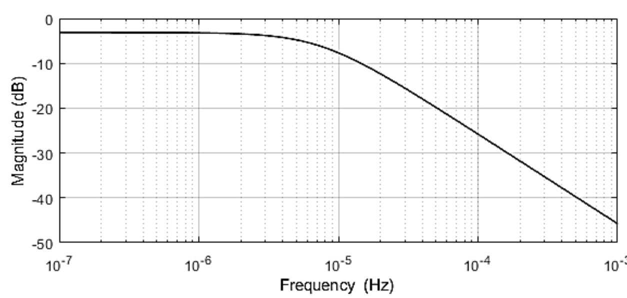

Figure 16. System magnitude response. Source: own.

The system phase response (92) and Figure 17.

$$\phi(\omega) = \frac{3{,}233\ x10^{-5}}{j\omega + 4{,}633\ x10^{-5}} = \infty \qquad (92)$$

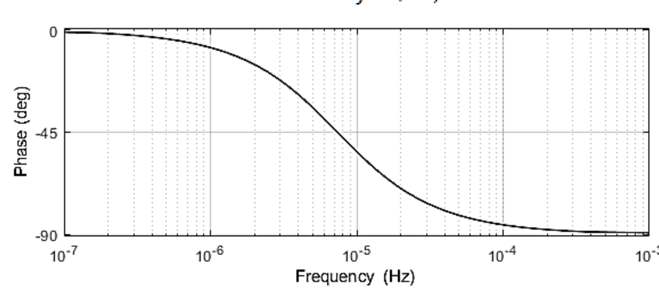

Figure 17 . System phase response. Source: own.

From the analysis performed (91) and (92), allows to see the non-stability of the system.

7. Conclusions

The dynamic systems model diverse physical phenomena through differential equations, the classic form to exert control on these systems is by means of the Laplace Transform, with this transform to carry differential equations in the time domain that in many cases are difficult to solve, it becomes simpler in the Frequency Domain, in this domain there are several tools that allow both modeling and control actions, within these tools is had a frequency analysis, transfer function, block diagrams among others.

In this paper has made a presentation of the Laplace Transform, its definition, properties and theorems. In addition, creates a framework of references in which it sees different uses of LT in research and highlights the importance of this in dynamic systems presenting its various uses in the control systems analysis. Additionally, shows a methodology for the LT application in dynamic systems and finally, a multipurpose reactor modeling exercise was carried out where the LT allowed to find the model and other characteristics of the same one. It is important to emphasize that the referential framework used is oriented mainly towards the dynamic systems and control systems, where 39.18% of references speak of these thematic ones; in subjects related to electricity, electronics and telecommunications was obtained 17.56% of references; 33.78% speak of definition, properties and theorems of the Laplace Transform and 4.05% speak of the LT use in fields other than engineering, is to highlight that the research can be extended to the uses and advantages that the LT can provide in other fields other than those presented here in order to enrich this research.

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